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Non local theories: new rules for old diagrams

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Abstract: We show that a general variant of the Wick theorems can be used to reduce the time ordered products in the Gell-Mann & Low formula for a certain class on non local quantum field theories, including the case where the interaction lagrangian is defined in terms of twisted products.
The only necessary modification is the replacement of the Stueckelberg-Feynman propagator by the general propagator (the "contractor" of Denk and Schweda)

$$\mathscr{D}(y-y';\tau-\tau')=\frac{1}{i}\Big(\Delta_+(y-y')\theta(\tau-\tau')+\Delta_+(y'-y)\theta(\tau'-\tau)\Big),$$

where the violations of locality and causality are represented by the dependence of τ,τ' on other points, besides those involved in the contraction.
This leads naturally to a diagrammatic expansion of the Gell-Mann & Low formula, in terms of the same diagrams as in the local case, the only necessary modification concerning the Feynman rules. The ordinary local theory is easily recovered as a special case, and there is a one-to-one correspondence between the local and non local contributions corresponding to the same diagrams, which is preserved while performing the large scale limit of the theory.

Keywords: Renormalization Regularization and Renormalons, Field Theories in Higher Dimensions, Non-Commutative Geometry.

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Contents

1. Introduction

We consider the hamiltonian setup for perturbations of a single quantum neutral scalar field ϕ on the Minkowski spacetime, with a non local self interaction lagrangian of the form

$$\mathscr{L}_I(x) = \int dy_1 \cdots dy_n \, W_x(y_1, \dots, y_n) {:}\phi(y_1) \cdots \phi(y_n){:} \, . \tag{1.1}$$

If the kernel is chosen to fulfill

$$W_{x+a}(y_1, \dots, y_n) = W_x(y_1 - a, \dots, y_n - a) \, ,$$

the interaction hamiltonian $H_I(t) = -g \int d\boldsymbol{x} \, \mathscr{L}_I(t, \boldsymbol{x})$ is invariant under translations.

Motivations come from the quest for effective theories on classical spacetime, describing interactions on the quantum spacetime, as it was proposed by Doplicher, Fredenhagen and Roberts (DFR) in their seminal paper [1]; see also [2]. There, localization is described in terms of the covariant DFR quantum coordinates q^μ, whose commutators $i\lambda_P^2 Q^{\mu\nu} = [q^\mu, q^\nu]$ are constrained so to ensure stability of spacetime under localization (λ_P is the Planck length): no black holes should arise as an effect of localization *alone*. Note that, for this class of models, sharp localization still can be obtained in one coordinate, but not in all the coordinates simultaneously; pointwise localization is prevented by the uncertainty relations among the coordinates.

In particular, we consider the interaction

$$\mathscr{L}_I^Q(x) = \int_{\Sigma_1} d\sigma \, \mathscr{L}_I^\sigma(x) \, , \tag{1.2}$$

where

$$\mathscr{L}_I^\sigma(x) = {:}(\phi \star_\sigma \cdots \star_\sigma \phi)(x){:}\,, \tag{1.2'}$$

$\star_\sigma$ is the product twisted by the real antisymmetric matrix σ, and $d\sigma$ is the rotation invariant measure on a certain manifold Σ_1 of matrices, selected by optimal localization. The non pointwise nature of optimal localization is the source for the breakdown of the covariance of (1.2) under Lorentz boosts.[1] The lagrangian (1.2) is an effective interaction obtained by considering the quantization ${:}\phi(q)^n{:}$ of ${:}\phi(x)^n{:}$, and was first proposed in [1]. We recall that the DFR quantization *à la Weyl* of $\phi(x)$ is $\phi(q) = \int dk\, \check{\phi}(k) \otimes e^{ik_\mu q^\mu}$, where $\phi(x)$ is the ordinary neutral Klein–Gordon quantum field.

We also consider

$$\mathscr{L}_I^E(x) = \Phi(x)\,, \tag{1.3}$$

where $\Phi(x)$ is defined by

$$\Phi(\bar{q}) = E\{{:}\phi(q_1)\cdots\phi(q_n){:}\}\,.$$

Here, E is the quantum diagonal map introduced in [3], which generalizes the ordinary restriction $f\restriction_{\{x_j=x\}} = f(x,\ldots,x)$ to the case of functions $f(q_1,\ldots,q_n)$ of n independent quantum events, so to respect the quantum limitations on localizability (once again, minimizing the differences $q_i - q_j$ in the sense of optimal localization is responsible for the breakdown of full Lorentz covariance). Moreover, $\bar{q}^\mu$ are the "little" quantum coordinates of the mean position,[2] corresponding to Σ_1, and $\Phi(\bar{q}) = \int dk\, \check{\Phi}(k) \otimes e^{ik_\mu \bar{q}^\mu}$ is the DFR quantization of $\Phi(x)$ with respect to the coordinates $\bar{q}$.

The interactions $\mathscr{L}_I^Q$ and $\mathscr{L}_I^E$ both are covariant under translations and space rotations. The lagrangian $\mathscr{L}_I^E$ was found ultraviolet regular in [3].

Notwithstanding that it fails to be covariant even under space rotations, the lagrangian $\mathscr{L}_I^\sigma$ has also been extensively investigated on its own in the literature, motivated either by [1] or by later results in string theory. Our remarks also apply to this class of models, even when the matrix σ fails to be in Σ_1. In the latter case, however, the corresponding quantum coordinates fail to fulfill the DFR stability conditions (this is the case, e.g., of theories where the time coordinate commutes with all the space coordinates, $\sigma^{0j} = 0$).

The diagrammatic expansion of the Gell-Mann & Low formula for such interactions has been extensively investigated by several authors (see e.g. [4]). These and related questions were also carefully investigated in [5]. The price to pay for the breakdown of locality apparently were the need for a more elaborate topology of the diagrams than in the local case, and additional difficulties in dealing with the time ordering; see also [3].

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[1] Let σ be a real antisymmetric matrix in the joint spectrum Σ of the $Q^{\mu\nu}$'s; then there is a unique irreducible representation $i\lambda_P^2\sigma^{\mu\nu} = [q_\sigma^\mu, q_\sigma^\nu]$ of the DFR commutation relations by means of (non covariant) quantum coordinates q_σ^μ, and all regular irreducible representations arise in this way. Σ is the orbit of the standard symplectic matrix $\sigma_0 = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$ under the action of the full Lorentz group, while $\Sigma_1 \subset \Sigma$ is the orbit of σ_0 under the rotation subgroup. See [1].

[2] Note that the uncertainty of a coordinate $\bar{q}^\mu$ of the mean position of n independent quantum events is $1/\sqrt{n}$ times that of a single event; by "little", we mean that the joint spectrum of the operators $-i[\bar{q}_\mu, \bar{q}_\nu]$ is $(\lambda_P^2/n)\Sigma_1$ instead of the full manifold $(\lambda_P^2/n)\Sigma$. The coordinates $\bar{q}^\mu$ are not covariant under Lorentz boosts. See [3].

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An important step was taken in [6], where the possibility of treating the time ordering by means of single diagrams (instead of taking a different diagram for each arrangement of the "time stamps") was first considered.

Here, we observe that the algebra of the Wick theorems for the reduction of general time ordered products is essentially unaffected by the breakdown of locality. This allows a considerable simplification in the computations, which are basically the same as in the local case. It easily follows from this simple remark that the usual diagrams of the standard local perturbation theory can also be used in our case. We will obtain an alternative set of rules for ordinary diagrams, which contains the standard Feynman rules for the usual local theory as a special case. This complements the discussion of [7].

In the next section, we describe how to perform the Wick reduction of general time ordered products. In section 3, we use the general Wick theorems to obtain the diagrammatic expansion of the Gell-Mann & Low formula, and we show that the vacuum-vacuum parts cancel out. In section 4 we apply the framework to the motivating examples, and work out an example for the lagrangian (1.2'); in particular, we recover the modified rules for the modified diagrams already considered in the literature (we also shortly comment on the case where the Wick ordering is suppressed in (1.1)). We finally draw some conclusions in section 5.

2. General Wick theorems

We consider the situation where there is a one-to-one correspondence between k points $y_1, \dots, y_k$ of the space time and k parameters $\tau_1, \dots, \tau_k$, which we will call the times, for short (not to be confused with the time variables appearing in the free fields $\phi({y_j}^0, \boldsymbol{y}_j)$). Then, we can define the general time ordered product

$$T^{\{\tau_j\}}[\phi(y_1)\cdots\phi(y_k)] = \sum_{\pi} \phi(y_{\pi(1)})\cdots\phi(y_{\pi(k)})\theta(\tau_{\pi(1)} - \tau_{\pi(2)})\cdots\theta(\tau_{\pi(k-1)} - \tau_{\pi(k)})\,, \quad (2.1)$$

which, for any choice of the points y_j, coincides with the product of the $\phi(y_j)$'s taken in the order of decreasing times τ_j. The sum runs over all permutations π of k elements.

We make no assumptions on the τ_j's: they may be constants, as well as functions of the y_j's, or even depend on extra parameters.

Then, we define a general chronological contraction *with respect to the given correspondence* $y_j \leftrightarrow \tau_j$ as

$$:\phi(y_i)\cdots\underbrace{\phi(y_i)\cdots\phi(y_j)}\cdots\phi(y_n): \;=\; :\phi(y_1)\cdots\widehat{\phi(y_i)}\cdots\widehat{\phi(y_j)}\cdots\phi(y_n):\mathscr{D}(y_i - y_j; \tau_i - \tau_j)\,,$$

where a caret indicates omission, and

$$\mathscr{D}(x;\tau) = \frac{1}{i}\Big(\Delta_+(x)\theta(\tau) + \Delta_+(-x)\theta(-\tau)\Big) \quad (2.2)$$

is the general propagator, which was called contractor in [6]; here we prefer to emphasize that Wick reduction works as usual, up to putting "general" in front of everything.

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Indeed, with these only modifications, the Wick reduction of $T^{\{\tau_j\}}[\phi(y_1)\cdots\phi(y_k)]$ can be performed according to the same rules as in the local case ("first Wick theorem"); in particular, $T^{\{\tau_j\}}[\phi(y_1)\cdots\phi(y_k)]$ equals the sum of the terms which can be obtained by applying all possible choices of any number of general contractions (including no contraction at all) to $:\!\phi(y_1)\cdots\phi(y_k)\!:$.

The argument is exactly the same as in the original proof of Wick [9], which only relies on the fact that $[\phi(x),[\phi_+(x'),\phi_-(x'')]]=0$; the particular choice of a rule by which the times τ,τ',τ'' are associated to x,x',x'' plays no role. Here, we will content ourselves with checking the first Wick theorem in the simplest case: consider

$$T^{\{\tau,\tau'\}}[\phi(y)\phi(y')] = \phi(y)\phi(y')\theta(\tau-\tau') + \phi(y')\phi(y)\theta(\tau'-\tau)\,;$$

replacing

$$\begin{aligned}\phi(y)\phi(y') &= :\!\phi(y)\phi(y')\!: + \frac{1}{i}\Delta_+(y-y')\,,\\ \phi(y')\phi(y) &= :\!\phi(y')\phi(y)\!: + \frac{1}{i}\Delta_+(y'-y)\,,\end{aligned}$$

in the above expression, and using $:\!\phi(y)\phi(y')\!: = :\!\phi(y')\phi(y)\!:$, one obtains

$$T^{\{\tau,\tau'\}}[\phi(y)\phi(y')] = :\!\phi(y)\phi(y')\!: + \mathscr{D}(y-y';\tau-\tau') = :\!\phi(y)\phi(y')\!: + :\underbracket{\phi(y)\phi(y')}:\,,$$

indeed.

The ordinary time ordering and Wick theorems are reobtained as a special case, setting the time τ corresponding to y equal to the time component y^0 of y itself. Indeed, as already observed in [6],

$$\mathscr{D}(x;x^0) = \Delta_{SF}(x)$$

is the usual Stueckelberg-Feynman propagator [10, 11].

Since, by definition, $:\!\phi(x)\!: = \phi(x)$, then (2.1) is a special case of the general Wick "mixed product", where we closely follow the original terminology of Wick. We find it convenient to introduce the compact notations

$$\underline{y} = (y^1,\ldots,y^r)\,, \qquad \phi^{(r)}(\underline{y}) = \phi(y^1)\cdots\phi(y^r)\,, \tag{2.3}$$

so that the general Wick mixed product is defined by

$$\begin{aligned}&T^{\{\tau_j\}}[:\!\phi^{(r_1)}(\underline{y}_1)\!:\cdots:\!\phi^{(r_k)}(\underline{y}_k)\!:] =\\ &\quad= \sum_\pi :\!\phi^{(r_{\pi(1)})}(\underline{y}_{\pi(1)})\!:\cdots:\!\phi^{(r_{\pi(k)})}(\underline{y}_{\pi(k)})\!:\theta(\tau_{\pi(1)}-\tau_{\pi(2)})\cdots\theta(\tau_{\pi(k-1)}-\tau_{\pi(k)})\,;\end{aligned} \tag{2.4}$$

here all the points $y_j^1,\ldots,y_j^{r_j}$ belonging to the $j^{\rm th}$ monomial $:\!\phi^{(r_j)}(\underline{y}_j)\!:$ are made to correspond to the same time parameter τ_j.

The Wick reduction of a general Wick mixed product can be obtained similarly to the preceding case, where now the contractions between fields belonging to the same Wick monomials are forbidden ("second Wick theorem").

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See appendix A for a check of the second general Wick theorem in the simplest case, and the review [8] for the general proofs of all the above statements.

Of course, as in the ordinary case, matrix elements of general time ordered products may fail to make sense even as (tempered, say) distributions, and may well need renormalization; the existence of general time ordered products should be explicitly discussed for each choice of the times τ_j. Even when they exist, finite renormalization may well be necessary to perform the large scale limit.

3. Theme

Let

$$S = I + \sum_{N=1}^{\infty} \frac{i^N}{N!} \int dt_1 \dots dt_N \; T[H_I(t_1) \cdots H_I(t_N)]$$

be the Dyson series defined by the interaction hamiltonian $H_I(t)$ [12], where T indicates the standard Dyson time ordering prescription. We consider the Gell-Mann & Low formula

$$G_k(x_1, \dots, x_k) = \frac{\big\langle T[\phi(x_1) \cdots \phi(x_k) S]\big\rangle_0}{\langle S \rangle_0}\,, \tag{3.1}$$

for the Green functions of the interacting fields [13], where $\langle \cdot \rangle_0$ is the expectation on the free (Fock) vacuum, and the $\phi(x_j)$'s are free fields; we recall that here the Dyson time ordering prescription is extended so that, after expanding S in 3.1, at each order N the product of the operators $\phi(x_1), \dots, \phi(x_k), H_I(t_1), \dots, H_I(t_N)$ is taken in the order of decreasing $x_1^0, \dots, {x_k}^0, t_1, \dots, t_N$.

Following [1][3] we replace the hamiltonian $H_I(t) = -g \int d\boldsymbol{x}\; \mathscr{L}_I(t, \boldsymbol{x})$, where $\mathscr{L}_I$ is of the form (1.1), in the Gell-Mann & Low formula,

$$\begin{aligned} G_k(x_1, \dots, x_k) \;&=\; \frac{1}{\langle S \rangle_0} \sum_{N=0}^{\infty} \frac{(-ig)^N}{N!} \int dx_{k+1} \cdots dx_{k+N} \,\times \\ &\quad \times \big\langle T[\phi(x_1) \cdots \phi(x_k) \mathscr{L}_I(x_{k+1}) \cdots \mathscr{L}_I(x_{k+N})]\big\rangle_0 \end{aligned} \tag{3.2}$$

and we obtain

$$\begin{aligned} &G_k(x_1, \dots, x_k) = \\ &\quad = \frac{1}{\langle S \rangle_0} \sum_{N=0}^{\infty} \frac{(-ig)^N}{N!} \int dx_{k+1} \cdots dx_{k+N} \int d\underline{y}_1 \cdots d\underline{y}_N \; W_{x_{k+1}}(\underline{y}_1) \cdots W_{x_{k+N}}(\underline{y}_N) \,\times \\ &\qquad \times \big\langle T^{\{{x_j}^0\}}[\phi(x_1) \cdots \phi(x_k) {:}\phi^{(n)}(\underline{y}_1){:} \cdots {:}\phi^{(n)}(\underline{y}_N){:}]\big\rangle_0\,, \end{aligned} \tag{3.3}$$

where we use the notations (2.3).

We can now reduce the general mixed time ordered product which appears in (3.3), where the times driving the time ordering are $\tau_j = {x_j}^0$ for $j = 1, \dots, k+N$.

[3] See in particular the end of [1, section 6]. The Dyson series and the Gell-Mann & Low formula are purely quantum mechanical; the standard derivation of (3.2) is not affected by the breakdown of locality and causality. Non locality remains hidden in the definition of the Lagrangean density.

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If a general contraction involves two external fields $\phi(x_i), \phi(x_j)$, then the general propagator is $\Delta_{SF}(x_i - x_j)$; if it involves an external field $\phi(x_i)$ and a field $\phi(y^u_j)$ in a Wick monomial, then the general propagator is

$$\mathscr{D}(x_i - y^u_j; {x_i}^0 - {x_j}^0)$$

if it involves two fields $\phi(y^u_i)$ and $\phi(y^v_j)$ belonging to distinct Wick monomials, then we pick a

$$\mathscr{D}(y^u_i - y^v_j; {x_i}^0 - {x_j}^0)\,;$$

the remaining contractions are forbidden.

Due to the form of (1.1), the kernel W_x can be assumed to be totally symmetric in $y_1, \ldots, y_n$, for any x fixed (otherwise, we could symmetrize it without changing the interaction). One easily checks that, with the rules R1–R4 given below,

$$G_k(x_1, \ldots, x_k) = \frac{1}{\langle S \rangle_0} \sum_\gamma m(\gamma) I_\gamma(x_1, \ldots, x_k)\,,$$

where the sum runs over all the diagrams with k external points and any number of unlabeled vertices.[4] With $v(\gamma)$ the number of vertices of γ,

$$m(\gamma) = \frac{(n!)^{v(\gamma)}}{s(\gamma)}\,; \tag{3.4}$$

above, $s(\gamma)$ is the product of $\ell!$ over all unordered pairs of vertices, where ℓ is the number of lines connecting each pair.

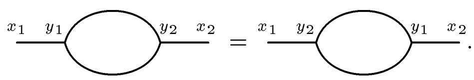
The contribution I_γ corresponding to the diagram γ (external points labeled by $x_1, \ldots, x_k$) is given by the integral of the expression obtained by means of the following rules:

R1 *label each vertex by an index j in $\{k+1, \ldots, k+v(\gamma)\}$, and otherwise arbitrary (different indices for different vertices); for each vertex j, take a factor*

$$-igW_{x_j}(y^1_j, \ldots, y^n_j)\,dx_j\,dy^1_j \cdots dy^n_j\,; \tag{3.5}$$

R2 *for each line connecting two external points, take as a factor the corresponding Stueckelberg-Feynman propagator;*

[4] We label the external points by $x_1, \ldots, x_k$, but we use diagrams with *unlabeled* vertices; this is the same as labeling the vertices, while declaring that, anyway,



Correspondingly, each diagram γ has multiplicity $v(\gamma)!m(\gamma)$. Note that if γ arises at order N in the perturbation series, then $N! = v(\gamma)!$.

R3 *for each line connecting the external point* x_i *and the* j^{th} *vertex, pick at random one of the unused* $y_j^1, \dots, y_j^n$*, say* y_j^u*, then take*

$$\mathscr{D}(x_i - y_j^u; {x_i}^0 - {x_j}^0)$$

and mark y_j^u *as used;*

R4 *for each line connecting the* i^{th} *and* j^{th} *vertices, pick at random two unused* y_i^u *and* y_j^v*, then take*

$$\mathscr{D}(y_i^u - y_j^v; {x_i}^0 - {x_j}^0)$$

and mark y_i^u, y_j^v *as used.*

The adiabatic switch and/or the restriction to finite volume are/is obtained as usual by turning the coupling constant g into a suitable function $g(x_j)$ in rule R1.

If we take $W_x(y_1, \dots, y_n) = \prod_j \delta(y_j - x)$, then (1.1) is the ordinary local lagrangian $:\phi(x)^n:$. By integrating the variables y_j out of (3.5), and correspondingly setting $y_j^u = x_j$ in rules R2, R3, we obtain the usual Feynman rules of the local perturbation theory as a special case.

It is straightforward to check that I_γ factorizes over the components of γ, so that in particular one can factorize the vacuum-vacuum parts. Since the combinatorics is the same as in the local case (the combinatorial factors do not depend on the choice of the kernels), it is natural to expect that the vacuum-vacuum parts cancel precisely with $1/\langle S\rangle_0 = (\sum_\gamma^{(0)} m(\gamma) I_\gamma)^{-1}$ by the same mechanism as in the local case (the sum $\sum_\gamma^{(0)}$ is restricted to the vacuum-vacuum diagrams, i.e. the diagrams with no external points). Indeed, by inspection of (3.4), one finds that $m(\gamma)$ factorizes over the connected components of γ; since the most general diagram γ is the disjoint union of a vacuum-vacuum diagram and a diagram with no vacuum-vacuum components, we obtain

$$G_k(x_1, \dots, x_k) = \sum_\gamma^{(1)} m(\gamma) I_\gamma(x_1, \dots, x_k)\,, \tag{3.6}$$

where the sum $\sum_\gamma^{(1)}$ is restricted to the diagrams with no vacuum-vacuum components. We thus reproduced in the present setting the original argument of the early days of local perturbation theory.

Equations representing the Fourier transform $\widehat{I}_\gamma(p_1, \dots, p_k)$ of $I_\gamma(x_1, \dots, x_k)$ are not so manageable; see however [6]. In the local case, the Stueckelberg–Feynman propagators are functions of differences of points; their Fourier transforms as functions of two variables then contain delta functions which make the convolutions trivial. Here this nice feature is lost for lines connecting an external point to a vertex; moreover, as a function of the difference $y_i^u - y_j^v$, the general propagator $\mathscr{D}(y_i^u - y_j^v; {x_i}^0 - {x_j}^0)$ connecting two vertices has Fourier transform depending parametrically on other variables; the resulting expression is rather cumbersome. One might observe that $\mathscr{D}(x - x'; \tau - \tau')$ depends on the difference $(x, \tau) - (x', \tau')$ of 5-vectors; however, we refrain from pursuing this idea, since it does not seem to provide any substantial simplification.

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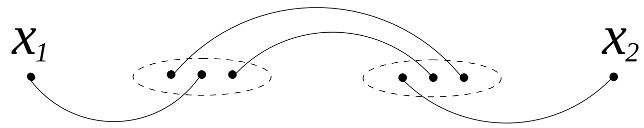
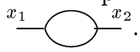


Figure 1: Fat diagram (in position space) of a second order contribution to $G_2(x_1, x_2)$; the points surrounded by a dashed line belong to the same fat vertex, and are ordered from left to right. Dashed lines only are a device to indicate sets of points, no topological information is attached to them. This connected diagram is of class .

4. Variations

The above rules are suitable for general discussions. On practical purposes, however, it may be useful to specialize the rules to the kernel which is actually considered, also for the sake of comparison with the existing literature.

4.1 First variation

For the lagrangian (1.3), the kernel W_x is of the form

$$W_x(y_1, \ldots, y_n) = \delta\big(x^0 - \tau(y_1, \ldots, y_n)\big) w(y_1, \ldots, y_n)\,, \tag{4.1}$$

where both τ and w are totally symmetric. See [3] for explicit expressions; τ is called there the "average time" of the lagrangian. This allows for integrating out the ${x_j}^0$'s ($j > k$). Correspondingly, we replace (3.5) by

$$-igw(y_j^1, \ldots, y_j^n) d\boldsymbol{x}_j dy_j^1 \cdots dy_j^n\,;$$

in rule R1, and we replace each time argument ${x_j}^0$ corresponding to a vertex (i.e. $j > k$) by $\tau(\underline{y}_j)$ in the general propagators of rules R3, R4.

The coupling constant g has to be turned into a function $g(\tau(\underline{y}_j), \boldsymbol{x}_j)$, to investigate the adiabatic switch and/or the finite volume theory.

4.2 Second variation

For n even, the kernel W_x for the lagrangian[5] (1.3) does not appear to have a compact expression for its totally symmetric part. Hence each integral has to be expanded into $m(\gamma)$ distinct integrals. In this case, it is perhaps more practical to let the diagrams take care of the symmetrization; moreover, one might wish to adhere to the philosophy that each diagram should correspond to one integral. This also will explain the reason why diagrams with more elaborate topologies may arise in some contexts.

We consider "fat diagrams" (see figure 1): a fat diagram $\vec{\gamma}$ consists of external points and "fat vertices"; a fat vertex is an *ordered* set of n points; each point originates exactly one line; no line can connect two points in the same fat vertex; fat vertices are unlabeled; two diagrams are the same if they can be made to coincide (by means of a smooth deformation

[5] Here, $W_x(y_1, \ldots, y_n) = \int_{\Sigma_1} d\sigma\; C(y_1 - x, \ldots, y_n - x)$, where the explicit form of the kernel C_n is given in eq.s (C.2) and (C.4) of [1, appendix C] (replace Q by σ in those equations).

in 3-space) so that the inner order of the points within each fat vertex is respected. These diagrams are essentially those already considered in the literature (see [3, 6]).

We may introduce an equivalence relation on fat diagrams; two fat diagrams $\vec{\gamma}, \vec{\gamma}'$ are equivalent if they shrink to the same ordinary diagram γ when we shrink the fat vertices to points (equivalently: if they are the same fat diagrams up to the inner order in the fat vertices). In this case, we say that $\vec{\gamma}, \vec{\gamma}'$ both are of class γ. By definition, a fat diagram $\vec{\gamma}$ is connected if it is of class γ with γ connected (equivalently: we say that two points in a fat diagram are connected by a chain of lines if discontinuities only take place within fat vertices; then $\vec{\gamma}$ is connected if any two of its points are connected by a chain of lines).

With these modifications,

$$G_k(x_1, \ldots, x_k) = \sum_{\vec{\gamma}}^{(0)} \frac{1}{s(\gamma)} J_{\vec{\gamma}}(x_1, \ldots, x_k) \,,$$

where $s(\gamma) = (n!)^{v(\gamma)}/m(\gamma)$, and we replace the rules R3, R4 by the rules R3', R4' below; note that rule R1 remains unchanged up to considering "vertex" as a synonym of "fat vertex", but now W_x may fail to be symmetric. Rules R3', R4' differ from R3, R4 in that we keep track of the inner order of each fat vertex, by associating the u^{th} variable y_j^u in $(y_j^1, \ldots, y_j^n)$ to the u^{th} point in the j^{th} fat vertex.

R3' *For each line connecting the external point x_i to the u^{th} point in the j^{th} vertex, take the factor*

$$\mathscr{D}(x_i - y_j^u; {x_i}^0 - {x_j}^0);$$

R4' *for each line connecting the u^{th} point in the i^{th} vertex to the v^{th} point in the j^{th} vertex, take the factor*

$$\mathscr{D}(y_i^u - y_j^v; {x_i}^0 - {x_j}^0).$$

Note that $J_{\vec{\gamma}}$ also factorizes over the connected components of $\vec{\gamma}$. With $[\vec{\gamma}]$ the equivalence class of $\vec{\gamma}$, we write $[\vec{\gamma}] = \gamma$ to say that $\vec{\gamma}$ is of class γ. Then setting

$$I_\gamma(x_1, \ldots, x_k) = \frac{s(\gamma)}{m(\gamma)} \sum_{\vec{\gamma} \in \gamma} J_{\vec{\gamma}}(x_1, \ldots, x_k) \,,$$

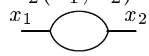
we recover the general expansion (3.6). Note also that, if W_y is totally symmetric, then $I_\gamma = J_{\vec{\gamma}}$ for each $\vec{\gamma}$ of class γ.

It is possible to extend the methods of Wick reduction of the Gell-Mann & Low formula to the case where the Wick ordering in (1.1) is suppressed. Note that, in that case, the kernel W_x cannot be symmetrized, so that we should resort to the fat diagrams of the present variation. Moreover, lines connecting different points y_j^u, y_j^v within the same fat vertex (oriented according to the inner order of fat vertices, namely from y_j^u to y_j^v if $u < v$), should be allowed in the fat diagrams; these additional lines would correspond to the ordinary two–points functions[6] $\frac{1}{i}\Delta_+(y_j^u - y_j^v)$. This way, one could recover the rules discussed in [6].

[6]To see this in a simple example, we replace $\phi(y)\phi(z) = {:}\phi(y)\phi(z){:} + \frac{1}{i}\Delta_+(y-z)$ into $\phi(x)\phi(y)\phi(z)\theta(\tau - \tau') + \phi(y)\phi(z)\phi(x)\theta(\tau' - \tau)$, obtaining $\phi(x)\frac{1}{i}\Delta_+(y-z) + T^{\{\tau,\tau'\}}[\phi(x){:}\phi(y)\phi(z){:}]$; by Wick reduction of the chronological product, the latter expression equals $\phi(x)\frac{1}{i}\Delta_+(y-z) + \phi(y)\mathscr{D}(x-z;\tau-\tau') + \phi(z)\mathscr{D}(x-y;\tau-\tau')$.

4.3 Third variation

For n odd, the kernel W_x corresponding to the lagrangian (1.2) is of the form (4.1), where none of w, τ is symmetric. Hence one could mix the flavours of the first and second variation, to write down the appropriate rules. We leave this to the reader, and we work out an example instead, using the variant (1.2') of the lagrangian (no integration over σ).

We consider the second order contribution to the two points function $G_2(x_1, x_2)$ with non local ϕ^3 self interaction, corresponding to the fat diagram $\vec{\gamma}$ (of class 

) of figure 1. Using the kernels[7] computed in [1, appendix C], we get

$$H_I(t) = -\frac{1}{\pi^4}\int dy_1 dy_2 dy_3 {:}\phi(y_1)\phi(y_2)\phi(y_3){:} e^{2i(y_1-y_3)\sigma^{-1}(y_2-y_3)}\delta^{(1)}(t-\tau)\,,$$

where

$$\tau(y_1, y_2, y_3) = {y_1}^0 - {y_2}^0 + {y_3}^0\,,$$

and $x\sigma^{-1}y = x^\mu(\sigma^{-1})_{\mu\nu}y^\nu$. The desired integral is then

$$\begin{aligned} J_{\vec{\gamma}}(x_1, x_2) = \frac{(ig)^2}{\pi^8}\int \left(\prod_{u=1}^{3} dy_1^u dy_2^u\right) & e^{2i(y_1^1-y_1^3)\sigma^{-1}(y_1^2-y_1^3)+2i(y_2^1-y_2^3)\sigma^{-1}(y_2^2-y_2^3)} \times \\ & \times \mathscr{D}(x_1 - y_1^2; {x_1}^0 - \tau(\underline{y}_1))\mathscr{D}(x_2 - y_2^1; {x_2}^0 - \tau(\underline{y}_2)) \times \\ & \times \mathscr{D}(y_1^1 - y_2^3; \tau(\underline{y}_1) - \tau(\underline{y}_2))\mathscr{D}(y_1^3 - y_2^2; \tau(\underline{y}_1) - \tau(\underline{y}_2))\,. \end{aligned}$$

5. Conclusions

We obtained a unified treatment of the Wick reduction of time ordered products and of the diagrammatic expansion of the Gell-Mann & Low formula for the non local lagrangians of interest to us; the latter is based on the usual diagrams, and contains the ordinary local Feynman rules as a special case. We also recovered from it the modified rules for the modified diagrams which are currently investigated in the literature.

Indeed, the non local lagrangians considered here are defined in terms of a length scale: the Planck length $\lambda_P \sim 1.6 \times 10^{-33}$cm. In the large scale limit, where the quantum nature of the spacetime is not directly visible any more, we expect to get back "diagramwise" to the ordinary (non renormalized) local ϕ^n model.

In principle, it should be possible to implement in position space a natural variant of the BPHZ renormalization procedure [14, 15, 16]; the R-operation is a rather general method for subtracting divergences, whose mechanism is quite insensitive to the particular prescription for the subtractions. Moreover, it might be possible to adapt the reformulation of the BPHZ procedure in terms of Hopf algebras, due to Kreimer [17].

Of course, a very delicate point is to detect the correct subtraction prescription.

We will discuss these issues in a forthcoming paper, under a different, equivalent point of view.

[7] With Q replaced by σ. Note that they only are meaningful when σ fulfill $\det\sigma \neq 0$, which is not the case of time/space commutativity. Explicit kernels for the case $\det\sigma = 0$ do not seem to be available in the literature, to the best of the author's knowledge.

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A. The second general Wick theorem

According to the second general Wick theorem, the general Wick mixed product (2.4) equals the sum of the terms obtained by applying all possible choices of any number (including none) of allowed general contractions to

$$:\overbrace{\phi(y_1^1)\phi(y_1^2)\cdots\phi(y_1^{r_1})}^{\tau_1}\ \overbrace{\phi(y_2^1)\phi(y_2^2)\cdots\phi(y_2^{r_2})}^{\tau_2}\ \cdots\ \overbrace{\phi(y_k^1)\phi(y_k^2)\cdots\phi(y_k^{r_k})}^{\tau_k}:,$$

where no contraction is allowed, which involves two fields associated with the same τ_j.

Since, once again, the proof is the same as in [9], we check this statement for the purpose of illustration in the case of the mixed product

$$T^{\{\tau,\tau'\}}[\phi(y){:}\phi(y_1')\phi(y_2'){:}] = \phi(y){:}\phi(y_1')\phi(y_2'){:}\theta(\tau-\tau') + {:}\phi(y_1')\phi(y_2'){:}\phi(y)\theta(\tau'-\tau)\,. \quad \text{(A.1)}$$

We have ($\phi^{\pm}$ are the creation and annihilation parts, as usual)

$$\begin{aligned}\phi(y){:}\phi(y_1')\phi(y_2'){:} &= {:}\phi^+(y)\phi(y_1')\phi(y_2'){:} + \phi^-(y){:}\phi(y_1')\phi(y_2'){:}\\ &= {:}\phi(y)\phi(y_1)\phi(y_2'){:} + \frac{1}{i}\Delta_+(y-y_1')\phi(y_2') + \frac{1}{i}\Delta_+(y-y_2')\phi(y_1)\,.\end{aligned}$$

Analogously,

$${:}\phi(y_1')\phi(y_2'){:}\phi(y) = {:}\phi(y_1)\phi(y_2')\phi(y){:} + \frac{1}{i}\Delta_+(y_1'-y)\phi(y_2') + \frac{1}{i}\Delta_+(y_2'-y)\phi(y_1)\,.$$

Replacing the above in (A.1), we obtain

$$\begin{aligned}T^{\{\tau,\tau'\}}[\phi(y){:}\phi(y_1')\phi(y_2'){:}] &= {:}\phi(y)\phi(y_1')\phi(y_2'){:} + \mathscr{D}(y-y_1')\phi(y_2') + \mathscr{D}(y-y_2')\phi(y_1')\\ &= {:}\phi(y)\phi(y_1')\phi(y_2'){:} + {:}\underbrace{\phi(y)\phi}(y_1')\phi(y_2'){:} + {:}\underbrace{\phi(y)\phi(y_1')\phi}(y_2'){:}\,;\end{aligned}$$

note that the (ill defined) general contraction ${:}\phi(y)\underbrace{\phi(y_1')\phi}(y_2'){:}$ is missing.

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